# 6 STRESS AND DEFORMATION

## 6.1 SMALL AND LARGE DEFORMATIONS

When a force is applied to a body then that body will be deformed. This is true for any material, including polymers. Figure 33 shows a rod that is deformed by a compression force (left) and by a tensile force (right). The original not deformed rod is shown in the middle.

We define the deformation or strain (ε) as the change of length of the body (Δl) divided by the original length of the body ($l_0$):

*Equation 22*

$$\varepsilon = \frac{\Delta l}{l_0}$$

Example: When a body with a length of 100 mm is stretched by 2 mm then the strain is 2 / 100 = 0.02.

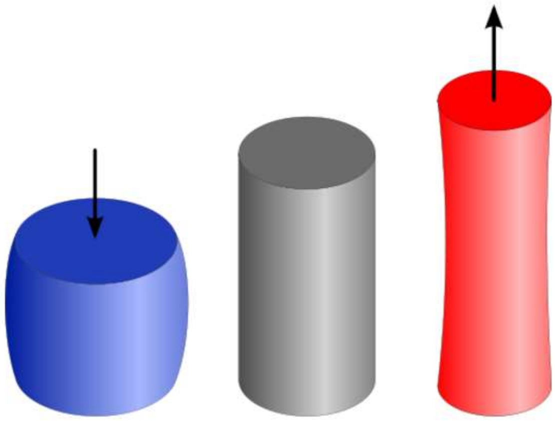

*Figure 33: A body subjected to a force will be deformed (commons.wikimedia.org).*

Instead of the force (F) it is common to use stress instead (σ), which is the force per unit area (A) in the body:

*Equation 23*

$$\sigma = \frac{F}{A}$$

Example: A force of 100 N on a body with a surface of 1000 $cm^2$ (= 0.1 $m^2$ ) will cause a stress of 100 / 0.1 = 1000 $N/m^2$, which is the same as 1000 Pa.

In case of a small deformation the stress is proportional to the deformation. This fact is known as Hooke's law. It says that the stress (σ) is equal to the deformation (ε) times the  Young's modulus (Y):

*Equation 24*

$$\sigma = Y\varepsilon$$

The Young's modulus is a material property. Example: Steel has an modulus of 210,000 MPa (1 MPa = $10^6$ $N/m^2$). In order to create a strain of 0.02 in the bar a stress equal to 210,000 x 0.01 = 2,100 MPa is needed.

With this set of equations the stresses and strains in many situations are well described as long as the strain is small: not more than something like 0.03. When a body is deformed more than a few percent than Hooke's law is not valid anymore.

Most polymers can be deformed much more than a few percent, especially when they are in the rubber phase. Think of a rubber band that can be stretched five to ten times before it breaks.

Therefore, we need another way to describe the relation between stress and deformation in polymers. It is common to specify the deformation of the polymer by the stretch ratio ($\lambda$), which is the new length (l) divided by the original length ($l_0$):

*Equation 25*

$$\lambda = \frac{l}{l_0}$$

Example: A body has an original length of 10 cm. If the body is stretched to a length of 15 cm then the stretch ratio is 15 / 10 = 1.5. If that same body would have been compressed to 5 cm then the stretch ratio is 5 / 10 = 0.5.

In case of large deformations, where Hooke's law is not valid, the relation between stress and stretch ratio is as follows:

*Equation 26*

$$\sigma = G\left(\lambda^2 - \frac{1}{\lambda}\right)$$

In this equation, G is the shear modulus of the body. Like the Young's modulus it is a material property. Usually the shear modulus is about 1/3 of the Young's modulus (G = Y/3).

Example: Rubber has a shear modulus of about 10 MPa. If the rubber is stretched 2 times then the stress in the rubber will be 10 x ($2^2$ – ½) = 35 MPa. In case of undeformed rubber the stretch ratio is 1 and the stress will be 10 x (1 – 1) = 0 MPa.

## 6.2 NEO-HOOKEAN MODEL FOR LARGE DEFORMATIONS

For describing the behaviour of a rubber we will use the "**Neo-Hookean model**". This model is an extension of Hooke's law for the case of large deformations. The model of neo-Hookean solid is usable for plastics and rubber-like substances. Since deformations (strains) and resulting true stresses are dependent on the direction, tensor notation will be used in this chapter.

The stress tensor σ is related to the shear modulus G of the polymer, the deformation gradient tensor **F** and the pressure p as follows:

*Equation 27*

$\boldsymbol{\sigma} = G\mathbf{F}\mathbf{F}^{\mathbf{T}} - p\mathbf{F}\mathbf{F}^{\mathbf{-1}}$ (or written in index notation: $\sigma_{ij} = G\sum_{k=1}^{3} F_{ik}F_{jk} - p\sum_{k=1}^{3} F_{ik}F_{kj}^{-1}$ )

σ is the stress tensor that describes the normal stresses and the shear stresses in the polymer:

*Equation 28*

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \text{with } \sigma_{ij} = \sigma_{ji}$$

The indices 1, 2 and 3 refer to the axis in a 3-dimensional space. For example, 1 could be the x-axis, 2 the y-axis and 3 the z-axis. The first index defines the plane on which the force is acting: 1 means the plane perpendicular to the x-axis, 2 the plane perpendicular to the y-axis and 3 the plane perpendicular to the z-axis. The second index defines the direction of the force: 1 in the direction of the x-axis, etc. $\sigma_{11}$, $\sigma_{22}$ and $\sigma_{33}$ are the normal stresses; $\sigma_{12}$, $\sigma_{13}$ and $\sigma_{23}$ are the shear stresses.

The deformation gradient tensor **F** describes all shear and strain deformations in the rubber material. It is the ratio between the deformed state (x) and the undeformed state (X):

*Equation 29*

$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

| In the case of a simple shear $\gamma = \gamma_{12}$ on plane 1 in axis direction 2 **F** becomes: $\mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | In the case of uniaxial extension $\lambda = \lambda_{11} = L_{new} / L_{old}$ on plane 1 in axis direction 1 **F** becomes: $\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & 1/\sqrt{\lambda} \end{bmatrix}$ |
|---|---|

$F^T$ is the transposed deformation tensor. This tensor is simply generated by exchanging the rows for the columns ($F_{ij} \leftrightarrow F_{ji}$).

| In case of a simple shear**:** $\mathbf{F}^{\mathbf{T}} = \begin{bmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | In case of uniaxial extension**:** $\mathbf{F}^{\mathbf{T}} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & 1/\sqrt{\lambda} \end{bmatrix}$ |
|---|---|

$F^{-1}$ is the inverse of the deformation tensor. By definition: $\mathbf{FF^{-1}} = I$, where I is the unit tensor with $I_{ij} = 1$ for i=j and $I_{ij} = 0$ for $i \neq j$.

| In case of a simple shear: | In case of uniaxial extension: |
|---|---|
| $\mathbf{F}^{-1} = \begin{bmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\mathbf{F}^{-1} = \begin{bmatrix} 1/\lambda & 0 & 0 \\ 0 & \sqrt{\lambda} & 0 \\ 0 & 0 & \sqrt{\lambda} \end{bmatrix}$ |

## 6.3 Example: Simple shear in rubber

Let us consider the case that a rubber is only sheared on plane 1 in the axis direction 2 by an amount of $\gamma$. Now Equation 27 becomes:

*Equation 30*

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left( G \begin{bmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - p \begin{bmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) =$$

$$\begin{bmatrix} G - p + G\gamma^2 & p\gamma + (G-p)\gamma & 0 \\ G\gamma & G - p & 0 \\ 0 & 0 & G - p \end{bmatrix}$$

Since the strain tensor is symmetric ($\sigma_{ij} = \sigma_{ji}$) it follows that p = G and we now find for the stress tensor:

*Equation 31*

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} G\gamma^2 & G\gamma & 0 \\ G\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sigma_{12}$ is the shear stress $\tau$. It increases linearly with the shear deformation $\gamma$. This is just like the well-known relation for shear strain and shear stress in an elastic solid according to Hooke's law for small deformations.

$\sigma_{11}$ is the normal stress that is induced in the rubber due to the shear deformation. It is also often referred to as the first normal stress difference ($\sigma_{11} - \sigma_{22}$). The normal stress increases quadratically with the shear deformation $\gamma$.

## 6.4 Example: Uniaxial extensional stress

Let us now consider a rubber of which plane 1 is stretched into the 1-direction. The stretch ratio $\lambda = L_{new} / L_{old}$. Now Equation 27 becomes:

*Equation 32*

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & 1/\sqrt{\lambda} \end{bmatrix} \left( G \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & 1/\sqrt{\lambda} \end{bmatrix} - p \begin{bmatrix} 1/\lambda & 0 & 0 \\ 0 & \sqrt{\lambda} & 0 \\ 0 & 0 & \sqrt{\lambda} \end{bmatrix} \right)$$

$$= \begin{bmatrix} G\lambda^2 - p & 0 & 0 \\ 0 & \dfrac{G}{\lambda} - p & 0 \\ 0 & 0 & \dfrac{G}{\lambda} - p \end{bmatrix}$$

Under uniaxial extension the stress in the 11-direction is $\sigma$ and the stress in all other directions is 0. It then follows that $p = G/\lambda$ and we find for the stress tensor:

*Equation 33*

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} G\left(\lambda^2 - \dfrac{1}{\lambda}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The stretch ratio $\lambda$ can also be expressed in strain $\varepsilon$ as $\lambda = e^{\varepsilon}$, which results in:

*Equation 34*

$$\sigma = G\left(e^{2\varepsilon} - e^{-\varepsilon}\right)$$

For small values of the strain ($\varepsilon << 1$) it then follows that $\sigma = 3G\varepsilon$ and for incompressible materials the Young's modulus Y = 3G. So for small deformations we find Hooke's law again:

*Equation 35*

$\sigma = Y\varepsilon$ (for $\varepsilon << 1$)